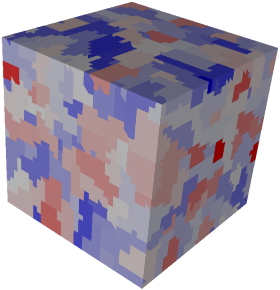
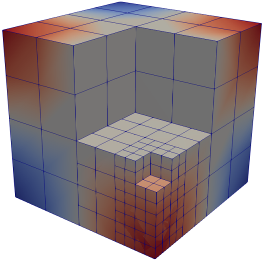
Hybrid polyhedral approximation of div-curl systems



Simon Lemaire
(INRIA, Univ. Lille)



ERC SyG NEMESIS, Kick-off workshop, Montpellier
June 20, 2024

Industrial context (1/2)

Nuclear safety

- Thermally-constrained metallic components: with aging, possible formation of cracks (stress corrosion cracking)
- Non-invasive detection of shallow flaws: based on eddy current testing (ECT)

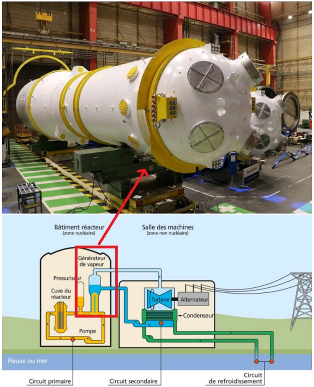


Figure: One of the 4 steam generators of an EPR plant (25m high, 510 tons).

Industrial context (2/2)

Numerical simulation of ECT

- Forward simulator: employed to calibrate/qualify ECT probes (*make the measurements fit the simulations*)
- Inverse simulator: employed to unravel the anatomy of flaws (*make the simulations fit the measurements*)

Forward model

Find $\boldsymbol{e} : \Omega \to \mathbb{C}^3$ s.t.

$$\begin{cases} \mathbf{curl}(\mu^{-1}\mathbf{curl}\, \boldsymbol{e}) + i\omega\sigma\boldsymbol{e} = -i\omega\boldsymbol{j} & \text{in } \Omega, \\ \mathrm{div}(\varepsilon\boldsymbol{e}) = 0 & \text{in } \Omega_{\mathrm{c}}^{c}, \\ \boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases}$$

with electric conductivity

$$\sigma = \begin{cases} 0 & \text{in } \Omega_{\mathrm{c}}^{c} \\ \sigma_{\mathrm{c}} & \text{in } \Omega_{\mathrm{c}} \end{cases}.$$

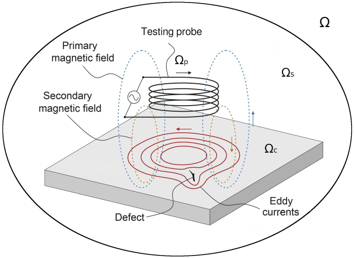


Figure: Sketch of a prototypical ECT setting.

Main numerical challenges

- Accurate approximation of the control signal $\rightsquigarrow$ high-order/enriched methods
- Modeling of the defects and 3D (re)meshing $\rightsquigarrow$ nonconforming/general meshes

Numerical simulation of ECT

- Forward simulator: employed to calibrate/qualify ECT probes (*make the measurements fit the simulations*)
- Inverse simulator: employed to unravel the anatomy of flaws (*make the simulations fit the measurements*)

Forward model

Find $\boldsymbol{e} : \Omega \to \mathbb{C}^3$, $\langle \varepsilon \boldsymbol{e}_{|\Omega_c^c} \cdot \boldsymbol{n}_c, 1 \rangle_{\partial\Omega_c} = 0$, s.t.

$$\begin{cases} \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{e}) + i\omega\sigma\boldsymbol{e} = -i\omega\boldsymbol{j} & \text{in } \Omega, \\ \operatorname{div}(\varepsilon\boldsymbol{e}) = 0 & \text{in } \Omega_c^c, \\ \boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases}$$

with electric conductivity

$$\sigma = \begin{cases} 0 & \text{in } \Omega_c^c \\ \sigma_c & \text{in } \Omega_c \end{cases}.$$

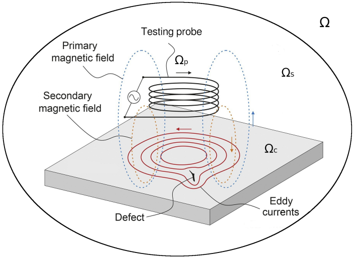


Figure: Sketch of a prototypical ECT setting.

Main numerical challenges

- Accurate approximation of the control signal ⇝ high-order/enriched methods
- Modeling of the defects and 3D (re)meshing ⇝ nonconforming/general meshes

Toy model

Let $\mathcal{D} \subset \mathbb{R}^3$ denote an open, bounded, connected, Lipschitz polyhedral domain. Recall the definition of Betti numbers:

- $\beta_0(\mathcal{D}) = 1$ (number of connected components of $\mathcal{D}$) and $\beta_3(\mathcal{D}) = 0$;
- $\beta_1(\mathcal{D})$: number of tunnels crossing through $\mathcal{D}$;
- $\beta_2(\mathcal{D})$: number of voids encapsulated into $\mathcal{D}$.

Figure: Betti numbers: (1,1,0,0).

Figure: Betti numbers: (1,0,1,0).

Magnetostatics

Given a current density $\boldsymbol{j} : \mathcal{D} \to \mathbb{R}^3$ satisfying $\operatorname{div} \boldsymbol{j} = 0$ in $\mathcal{D}$ and $\boldsymbol{j} \cdot \boldsymbol{n} = 0$ on $\partial\mathcal{D}$, find the magnetic field $\boldsymbol{h} : \mathcal{D} \to \mathbb{R}^3$ such that

$$\begin{cases} \mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j} & \text{in } \mathcal{D}, \\ \operatorname{div} \boldsymbol{b} = 0 & \text{in } \mathcal{D}, \\ \boldsymbol{h} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\mathcal{D}, \end{cases} \tag{$\mathfrak{P}_{\boldsymbol{\tau}}$}$$

with constitutive law $\boldsymbol{b} = \mu \boldsymbol{h}$, where $\mu \in \mathbb{R}_+^\star$ is the magnetic permeability.

Adjoint de Rham complex

$$\{0\} \xrightarrow{0} H_0^1(\mathcal{D}) \xrightarrow{-\,\mathbf{grad}} \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D}) \xrightarrow{\mathbf{curl}} \boldsymbol{H}_0(\mathrm{div};\mathcal{D}) \xrightarrow{-\,\mathrm{div}} L^2(\mathcal{D}) \xrightarrow{0} \{0\}$$

with homology spaces:

- $\mathfrak{H}^3 := \mathrm{Ker}(\mathbf{grad})/\mathrm{Im}(0) = \{0\}$ of dimension $\beta_3(\mathcal{D}) = 0$;
- $\mathfrak{H}^2 := \mathrm{Ker}(\mathbf{curl})/\mathrm{Im}(\mathbf{grad}) = \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl}^{\mathbf{0}};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div}^0;\mathcal{D})$ of dimension $\beta_2(\mathcal{D})$;
- $\mathfrak{H}^1 := \mathrm{Ker}(\mathrm{div})/\mathrm{Im}(\mathbf{curl}) = \boldsymbol{H}(\mathbf{curl}^{\mathbf{0}};\mathcal{D}) \cap \boldsymbol{H}_0(\mathrm{div}^0;\mathcal{D})$ of dimension $\beta_1(\mathcal{D})$;
- $\mathfrak{H}^0 := \mathrm{Ker}(0)/\mathrm{Im}(\mathrm{div})$ of dimension $\beta_0(\mathcal{D}) = 1$.

Fredholm alternative for $(\mathfrak{P}_{\boldsymbol{\tau}})$

Recall that the source current density satisfies $\boldsymbol{j} \in \boldsymbol{H}_0(\mathrm{div}^0;\mathcal{D})$.

- If $\beta_1(\mathcal{D}) = \beta_2(\mathcal{D}) = 0$, there exists a unique solution to $(\mathfrak{P}_{\boldsymbol{\tau}})$.
  - $\rightsquigarrow$ $\beta_1(\mathcal{D}) = 0 \rightsquigarrow \boldsymbol{H}_0(\mathrm{div}^0;\mathcal{D}) = \mathbf{curl}\big(\boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D})\big) \rightsquigarrow$ existence
  - $\rightsquigarrow$ $\beta_2(\mathcal{D}) = 0 \rightsquigarrow \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl}^{\mathbf{0}};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div}^0;\mathcal{D}) = \{\mathbf{0}\} \rightsquigarrow$ uniqueness
- If $\beta_1(\mathcal{D}) > 0$ or/and $\beta_2(\mathcal{D}) > 0$, a necessary condition of existence of a solution to $(\mathfrak{P}_{\boldsymbol{\tau}})$ is $\boldsymbol{j} \perp \mathfrak{H}^1$, then the solution is unique up to an element of $\mathfrak{H}^2$.

Weak form of ($\mathfrak{P}_{\boldsymbol{\tau}}$) [Kikuchi; 89]

Given $\boldsymbol{j} \in \boldsymbol{H}_0(\mathrm{div}^0;\mathcal{D})$, $\boldsymbol{j} \perp \mathfrak{H}^1$, find $(\boldsymbol{h}, p) \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D}) \times H^1_{\partial\mathcal{D}}(\mathcal{D})$ s.t.

$$\begin{cases} \displaystyle\int_{\mathcal{D}} \mathbf{curl}\,\boldsymbol{h}\cdot \mathbf{curl}\,\boldsymbol{v} + \mu \int_{\mathcal{D}} \boldsymbol{v}\cdot \mathbf{grad}\, p = \int_{\mathcal{D}} \boldsymbol{j}\cdot \mathbf{curl}\,\boldsymbol{v} & \forall \boldsymbol{v} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D}), \\ \displaystyle -\mu \int_{\mathcal{D}} \boldsymbol{h}\cdot \mathbf{grad}\, q = 0 & \forall q \in H^1_{\partial\mathcal{D}}(\mathcal{D}). \end{cases} \tag{$\mathsf{P}_{\boldsymbol{\tau}}$}$$

Remark that $p \equiv 0$ (test with $\boldsymbol{v} = \mathbf{grad}\, p \in \mathbf{grad}\left(H^1_{\partial\mathcal{D}}(\mathcal{D})\right) \subset \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl}^{\mathbf{0}};\mathcal{D})$).

Weber inequalities

- Weber inequalities are named after Christian Weber [Weber; 80].
- They are generalizations of the Poincaré inequality to the case of vector fields belonging to $\boldsymbol{H}(\mathbf{curl};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div};\mathcal{D})\left(\supset \boldsymbol{H}^1(\mathcal{D})\right)$, and featuring on $\partial\mathcal{D}$ either vanishing tangential trace (first) or vanishing normal trace (second).
- Ex.: First Weber inequality for $\beta_2(\mathcal{D}) = 0$: $\forall \boldsymbol{v} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div};\mathcal{D})$,

$$\|\boldsymbol{v}\|_{0,\mathcal{D}} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\mathcal{D}} + \|\,\mathrm{div}\,\boldsymbol{v}\|_{0,\mathcal{D}}.$$

Let $(\mathcal{T}_h, \mathcal{F}_h)$ be a polyhedral mesh of $\mathcal{D} \subset \mathbb{R}^3$, and $\ell \in \mathbb{N}$ a given polynomial degree.

Cell-wise polynomial decomposition

For $T \in \mathcal{T}_h$, let $\boldsymbol{x}_T$ be some point inside T such that T contains a ball centered at $\boldsymbol{x}_T$ of radius comparable to h_T. There holds

$$\mathcal{P}^\ell(T)^3 =: \boldsymbol{\mathcal{P}}^\ell(T) = \boldsymbol{\mathcal{G}}^\ell(T) \oplus \boldsymbol{\mathcal{P}}^{\ell-1}(T)\times(\boldsymbol{x} - \boldsymbol{x}_T),$$

where $\boldsymbol{\mathcal{G}}^\ell(T) := \mathbf{grad}\left(\mathcal{P}^{\ell+1}(T)\right)$, and the polynomial space $\boldsymbol{\mathcal{P}}^{\ell-1}(T)\times(\boldsymbol{x} - \boldsymbol{x}_T)$ is the so-called Koszul complement.

Face-wise polynomial decomposition

For $F \in \mathcal{F}_h$, let $\boldsymbol{x}_F$ be some point inside F such that F contains a disk centered at $\boldsymbol{x}_F$ of radius comparable to h_F. There holds

$$\mathcal{P}^\ell(F)^2 =: \boldsymbol{\mathcal{P}}^\ell(F) = \boldsymbol{\mathcal{R}}^\ell(F) \oplus \mathcal{P}^{\ell-1}(F)(\boldsymbol{x} - \boldsymbol{x}_F),$$

where $\boldsymbol{\mathcal{R}}^\ell(F) := \left(\mathbf{grad}_F\left(\mathcal{P}^{\ell+1}(F)\right)\right)^\perp$, with $\boldsymbol{z}^\perp$ the rotation of angle $-\frac{\pi}{2}$ of $\boldsymbol{z}$ in the oriented hyperplane H_F, and $\mathcal{P}^{\ell-1}(F)(\boldsymbol{x} - \boldsymbol{x}_F)$ is the Koszul complement.

$\rightsquigarrow$ For all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, there holds $\boldsymbol{\mathcal{G}}^\ell(T)_{|F}\times\boldsymbol{n}_F = \boldsymbol{\mathcal{R}}^\ell(F)$.

$\boldsymbol{H}(\mathbf{curl})$-like hybrid space

For $\ell \in \mathbb{N}$, we consider the hybrid space of unknowns

$$\underline{\boldsymbol{X}}_h^\ell := \left\{ \underline{\boldsymbol{v}}_h := \left((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_{F,\boldsymbol{\tau}})_{F \in \mathcal{F}_h} \right) \ : \ \begin{array}{ll} \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^\ell(T) & \forall T \in \mathcal{T}_h \\ \boldsymbol{v}_{F,\boldsymbol{\tau}} \in \boldsymbol{\mathcal{R}}^\ell(F) & \forall F \in \mathcal{F}_h \end{array} \right\},$$

endowed with the semi-norm

$$|\underline{\boldsymbol{v}}_h|^2_{\mathbf{curl},h} := \sum_{T \in \mathcal{T}_h} \left(\|\,\mathbf{curl}\,\boldsymbol{v}_T\|^2_{0,T} + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|\boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{v}_{T|F} \times \boldsymbol{n}_F) - \boldsymbol{v}_{F,\boldsymbol{\tau}}\|^2_{0,F} \right).$$

For $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_h^\ell$, we let $\boldsymbol{v}_h \in \boldsymbol{\mathcal{P}}^\ell(\mathcal{T}_h)$ be such that $\boldsymbol{v}_{h|T} := \boldsymbol{v}_T$ for all $T \in \mathcal{T}_h$.

Is $|\cdot|_{\mathbf{curl},h}$ a norm on a div-free subset of $\underline{\boldsymbol{X}}_{h,\mathbf{0}}^\ell := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_h^\ell \mid \boldsymbol{v}_{F,\boldsymbol{\tau}} \equiv \mathbf{0} \ \forall F \in \mathcal{F}_h^\partial \right\}$?

First hybrid Weber inequality (for $\beta_2(\mathcal{D}) = 0$) [Chave, Di Pietro, SL; 22]

For any $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{h,\mathbf{0}}^\ell$ s.t. $\int_{\mathcal{D}} \boldsymbol{v}_h \cdot \mathbf{grad}\, q = 0$ for all $q \in H^1_0(\mathcal{D})$,

$$\|\boldsymbol{v}_h\|_{0,\mathcal{D}} \lesssim |\underline{\boldsymbol{v}}_h|_{\mathbf{curl},h}.$$

Let $k \in \mathbb{N}^\star$ be a given polynomial degree. Define

$$\begin{aligned} A_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) &:= \int_{\mathcal{D}} \mathbf{curl}_h \boldsymbol{u}_h \cdot \mathbf{curl}_h \boldsymbol{v}_h \\ &\quad + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} \int_F [\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{u}_{T|F} \times \boldsymbol{n}_F) - \boldsymbol{u}_{F,\boldsymbol{\tau}}] \cdot [\boldsymbol{\pi}^k_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{v}_{T|F} \times \boldsymbol{n}_F) - \boldsymbol{v}_{F,\boldsymbol{\tau}}], \\ B_h(\underline{\boldsymbol{u}}_h, \underline{q}_h) &:= \int_{\mathcal{D}} \boldsymbol{u}_h \cdot \boldsymbol{G}^k_h(\underline{q}_h), \\ N_h(\underline{r}_h, \underline{q}_h) &:= \int_{\mathcal{D}} r_h q_h + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \int_F r_F q_F. \end{aligned}$$

Discrete problem (for $\beta_2(\mathcal{D}) = 0$)

Find $(\underline{\boldsymbol{h}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{h,\mathbf{0}} \times \underline{Y}^k_{h,0}$ such that

$$\begin{cases} A_h(\underline{\boldsymbol{h}}_h, \underline{\boldsymbol{v}}_h) + \mu B_h(\underline{\boldsymbol{v}}_h, \underline{p}_h) = \int_{\mathcal{D}} \boldsymbol{j} \cdot \mathbf{curl}_h \boldsymbol{v}_h & \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{h,\mathbf{0}}, \\ -\mu B_h(\underline{\boldsymbol{h}}_h, \underline{q}_h) + N_h(\underline{p}_h, \underline{q}_h) = 0 & \forall \underline{q}_h \in \underline{Y}^k_{h,0}. \end{cases}$$

The discrete problem has a unique solution satisfying

$$\left(|\underline{\boldsymbol{h}}_h|^2_{\mathbf{curl},h} + \|\underline{p}_h\|^2_{0,h} \right)^{1/2} \leq \|\boldsymbol{j}\|_{0,\mathcal{D}},$$

where $\|\underline{q}_h\|^2_{0,h} := N_h(\underline{q}_h, \underline{q}_h)$.

Convergence

Energy-error estimate [Chave, Di Pietro, SL; 22]

Assume that $\boldsymbol{j} \in \mathbf{curl}\left(\boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D})\right)$, and that $\beta_2(\mathcal{D}) = 0$. Suppose, in addition, that $\boldsymbol{h} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D})$ further satisfies $\boldsymbol{h} \in \boldsymbol{H}^{k+1}(\mathcal{T}_h)$. Then,

$$\left(|\underline{\boldsymbol{h}}_h - \underline{\boldsymbol{I}}_h^k(\boldsymbol{h})|^2_{\mathbf{curl},h} + \|\underline{p}_h\|^2_{0,h}\right)^{1/2} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2k} |\boldsymbol{h}|^2_{k+1,T}\right)^{1/2}.$$

- convergence of order $k \geq 1$ of $\|\mathbf{curl}_h \boldsymbol{h}_h - \mathbf{curl}\,\boldsymbol{h}\|_{0,\mathcal{D}}$
- observed convergence of order $k+1$ of $\|\boldsymbol{h}_h - \boldsymbol{h}\|_{0,\mathcal{D}}$ for $\mathcal{D}$ convex
- in practice, local elimination of all (magnetic and pressure) cell unknowns
- in the matching tetrahedral case, N_h can be removed

Numerical illustration

Academic test-case: $\mathcal{D} := (0,1)^3$, with $\mu = 1$ and exact solution

$$\boldsymbol{h}(x,y,z) = \big(\cos(\pi y)\cos(\pi z), \cos(\pi x)\cos(\pi z), \cos(\pi x)\cos(\pi y)\big)$$

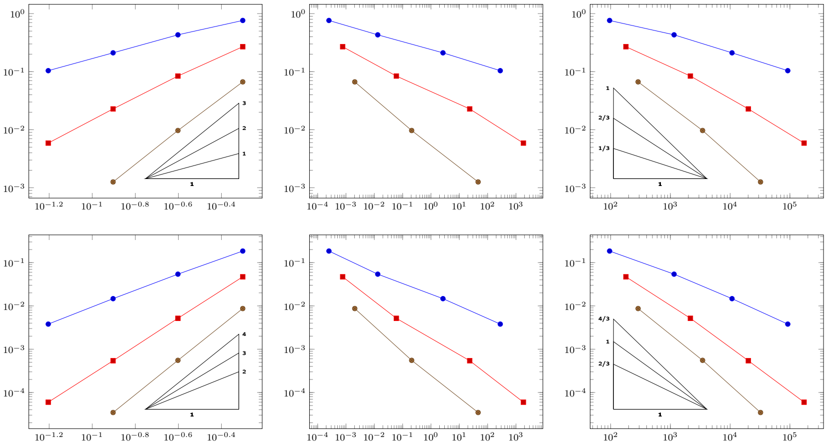


Figure: Relative energy-error (*top row*) and L^2-error (*bottom row*) vs. meshsize h (*left*), solution time in s (*center*), and #dof (*right*) on cubic meshes for $k \in \{1, 2, 3\}$.

Trivial topology

F. Chave, D. A. Di Pietro, and SL

A discrete Weber inequality on three-dimensional hybrid spaces with application to the HHO approximation of magnetostatics

M3AS, 2022

Nontrivial topology

SL and S. Pitassi

Discrete Weber inequalities and related Maxwell compactness for hybrid spaces over polyhedral partitions of domains with general topology

Found. Comput. Math., 2024

J. Dalphin, J.-P. Ducreux, SL, and S. Pitassi

Hybrid high-order approximation of div-curl systems on domains with general topology

In preparation

QUESTIONS?